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Consensus of Multi-agents in Switching Networks Using Input-to-State Stability of Switched Systems

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***Abstract*—Solvability of a consensus problem for multi-agent systems in switching networks heavily relies on stablization techniques for switched systems. The consensus problem becomes challenging when the agent dynamics are complicated that require stability of a switched system composed of unstable subsystems. Moreover, complicated dynamics bring other coupling structures into a switched system. To solve the consensus problem in such a scenario, we study the input-to-state stability property of a switched system with external influence and hence design a stablization controller for a switched system of interconnected components using the small gain theorem. The controller successfully leads to the solution to the consensus problem for a class of nonlinear, heterogeneous, and uncertain multi-agent systems in switching networks with jointly connected topology.**

***Index Terms*—Input-to-state stability (ISS), Switched systems, Consensus, Multi-agent systems (MAS), Small gain theorem, Nonlinear systems.**

I. Introduction

In recent years, many efforts have been put into developing control algorithms for multi-agent systems (MAS) of complicated dynamics, including nonlinear, heterogeneous, and uncertain dynamics; see [1], [2], [3], [4] for various settings. However, these results were achieved in a relatively simple network, in particular, of a fixed and connected topology. In many practical scenarios, the topology of communication network is not always fixed due to various factors such as external interference and hardware limitations. Therefore consensus of MASs in a time-varying network has attracted many attentions.

Technically, consensus of MASs in a switching network (that is, network topology and communication weights are switched over a finite set) essentially relies on the development of stabilization technique for switched systems. First of all, a relatively simple situation is that at least one topology in the finite set is connected. This situation corresponds to a switched system that consists of at least one stable subsystem and many effective switching algorithms can apply as discussed in the previous subsection. Along this research line, we can see works [5], [6], [7] under the assumption that the switching topologies are always connected and [8], [9] where at least one topology is assumed to be connected.

The main theme of this paper is a switching network but with time-varying topologies that may be unconnected during the entire time course. It is shown in the existing literature that a static controller can solve the consensus problem for some MASs under the so-called jointly connected assumption that the union of topologies during certain time intervals is connected, specifically, for linear MASs with (first order) single-integrator dynamics [10], [11].

A commonly used technique for handling the consensus problem of an MAS is to convert it into a stabilization problem for an error system between the MAS and the agreed dynamics. In a switching network, the resulting stabilization is formulated for a switched system. In particular, when none of the topologies is assumed to be connected, none of the subsystems of the switched system is assumed to be asymptotically stable. Nevertheless, it can be proved that, for a first-order linear MAS with a static controller, the corresponding subsystems are always at least *marginally stable*. In other words, the subsystem matrices contain eigenvalues of negative real parts and zero eigenvalues whose algebraic multiplicity is equal to the geometric multiplicity. This property enables a non-increasing Lyapunov function along each subsystem and hence guarantees system stability and consensus under the jointly connected condition.

However, for MASs with (second-order) double-integrator dynamics, the situation becomes essentially more difficult caused by complication of zero eigenvalues. In particular, the resulting error system for an unconnected network may contain zero eigenvalues whose algebraic multiplicity is strictly larger than the geometric multiplicity. Therefore, the stability analysis is conducted on a switched system for possibly *polynomially unstable* subsystems of divergent modes caused by repeated eigenvalues on imaginary axis. This phenomenon brings significant challenges that have yet to be completely overcome.

A successful consensus controller for MASs with double-integrator dynamics was given in [12] and [13] for instance, where a special zero velocity consensus scenario was considered. For this special case, the resulting switched system still consists of marginally stable subsystems. In other words, the difficulty of controlling a switched system with unstable subsystems was avoided. Following this direction, consensus of more general linear MASs was studied in [14] and [15], not surprisingly, also for marginally stable dynamics.

Therefore how to achieve consensus for double-integrator MASs in a general setting becomes challenging as it heavily relies on the stabilization technique for a switched system of unstable subsystems. One approach can be found in [16] and [17] where an additional constraint is needed that the topologies are switched rapidly and jointly connected. However, rapid network switching may cause many practical issues. Another approach was given in [18] but using a state dependent switching strategy. State measurement of the global network information for determining a switching strategy is impractical for a decentralized controller.

It is worth mentioning that switching topology has also been studied for cluster consensus. For example, for first-order linear MASs, cluster consensus is achieved under any switching strategy if some LMIs are satisfied [19]. But the solvability of the LMIs has yet to be discussed. Results on higher-order linear MASs were obtained in [20] based on an averaging method under sufficiently fast switching topology.

A more artistic design was given in [21] using the idea of dynamic coupling. The consensus problem unavoidably leads to stabilization of a switched system of unstable (especially polynomially unstable) subsystems. However, it was observed that each polynomially unstable subsystem matrix can be regarded as interconnection of two marginally stable matrices. As a result, the results for a switched system with marginally stable subsystems can be applied. The idea of dynamic coupling was further extended in [22] to solve the consensus

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problem of linear MASs with heterogeneous dynamics.

Now, a question arises: is there a solution for consensus of more general MASs in switching networks? In particular, the following still remains an open problem. *For an MAS whose dynamics are nonlinear, heterogeneous and uncertain, and contain a nominal double-integrator, is there a consensus algorithm in a switching network with jointly connected topology (not sufficiently fast switching or state-dependent)?*

This consensus problem is challenging because all the existing approaches based on a switched system of marginally stable subsystems do not apply as the MAS under investigation contains double-integrator dynamics. As the system contains nonlinear, heterogeneous and uncertain dynamics, a nonlinear dynamic compensator becomes indispensable, which much complicates the interconnection of the closed-loop system. More specifically, each polynomially unstable system in this scenario cannot be regarded as interconnection of simply two marginally stable subsystems, but with extra interconnection from the nonlinear compensator. Therefore, the approach [21] does not apply either. However, it motivates the study on a switched system with external input (to represent the extra interconnection). It turns out to exactly rely on the ISS property of a switched system.

The ISS property of a linear switched system with input is guaranteed if the switched system without input is uniformly exponentially stable under arbitrary switching signal [23]. When a switched system involves both ISS subsystems and subsystems that are not ISS, stability conditions in terms of the length of activated time for the ISS subsystems and that for non-ISS ones were derived in, e.g., [24]. There are just a few results in the existing literature analyzing the ISS property of a switched system where none of the subsystems is assumed to be ISS, see for example [25], [26], [27], [28]. It was shown in [25], [26] that the switched system with input is ISS if the average of subsystems is ISS, under a sufficiently fast switching signal. In [27], relaxed results are obtained where the derivative of Lyapunov functions of subsystems can be indefinite. But it is hard to design a switching signal such that these relaxed conditions are satisfied. And in [28], each subsystem does not necessarily have the ISS property in the state space but only in some subregion of the state space. Then by a properly designed state-dependent switching signal, the interconnected system can be stabilized.

In this paper, motivated by the existing results on ISS property of switched systems, we bring a new ISS analysis tool for a switched system with external input where none of the subsystems is ISS (every subsystem has a marginally stable or even unstable system matrix) under a class of state-independent switching signals. The tool is thus used to design of a consensus controller for a class of complicated MASs in switching networks.

II. Problem Formulation

Consider the MAS consisting of N autonomous agents whose dynamics are described as follows,

$$\begin{aligned} \dot{p}_i &= v_i \\ \dot{v}_i &= f_i(v_i, w_i) + u_i,\ i = 1, \cdots, N \end{aligned} \tag{1}$$

with $p_i, v_i, u_i \in \mathbb{R}$. This system contains a (second-order) double-integrator model as a special case, but with the function $f_i(v_i, w_i)$ representing the heterogeneous nonlinearities in the presence of unknown system parameters $w_i \in \mathbb{R}^{l_i}$. Assume that $f_i(v_i, w_i)$ is continuously differentiable with $f_i(0, w_i) = 0$. As a result, the derivative of $f_i(v_i, w_i)$ is bounded by a nonlinear function, that is, $|\partial f_i(v_i, w_i)/\partial v_i| \leq \mathcal{F}_i(v_i)$ for all w_i in a compact set. Let $\mathcal{F}(v) = \mathrm{diag}\{\mathcal{F}_1(v_1), \cdots, \mathcal{F}_N(v_N)\}$.

Definition 2.1: The group of N agents (1) is said to achieve consensus if

$$\lim_{t\to\infty}[p_i(t) - p_0(t)] = 0,\ \lim_{t\to\infty}[v_i(t) - v_0] = 0,\ \forall i = 1, \cdots, N$$

for some $p_0(t)$ and v_0 satisfying the following virtual reference dynamics

$$\begin{aligned} \dot{p}_0 &= v_0 \\ \dot{v}_0 &= 0. \end{aligned} \tag{2}$$

Consensus can be achieved by effective network communication. We first give a brief description of graph of communication network as follows. In particular, an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ is considered in this paper where $\mathcal{V} = \{1, \cdots, N\}$ denotes a finite non-empty set of nodes and $\mathcal{E} = \{(i,j) : i, j \in \mathcal{V}, i \neq j\} \subset \mathcal{V} \times \mathcal{V}$ represents the set of edges. Each graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ is with the specified edge weight $a_{ii}(t) \equiv 0$ (no selfloop) and $a_{ij}(t) \geq 0$ for $i \neq j$. More specifically, $a_{ij} > 0$ for $(i,j) \in \mathcal{E}$ and $a_{ij} = 0$ for $(i,j) \notin \mathcal{E}$. Let L be the Laplacian associated with $\mathcal{G}$ and defined as follows: the (i,i)-element of L is $\sum_{j\neq i} a_{ij}$ and the (i,j)-element is $-a_{ij}$ $(i \neq j)$.

Consensus of the MAS (1) in a fixed network has been solved in [3]. This paper however is concerned about a more complicated scenario of time-varying networks. More specially, we denote the time-varying graphs as $\mathcal{G}_{\sigma(t)}$ explicitly depending on network switching signal $\sigma(t) : [t_0, +\infty) \mapsto \mathbb{P}$ for $\mathbb{P} := \{1, 2, \cdots, M\}$. Also, the edge weight is explicitly denoted as $a_{ij}^{\sigma(t)}$ and the associated Laplacian $L_{\sigma(t)}$.

Remark 2.1: For each network $i \in \mathbb{P}$, the graph $\mathcal{G}_i$ is undirected and the associated Laplacian L_i has the property $L_i \mathbf{1}_N = 0$ and $\mathbf{1}_N^{\mathrm{T}} L_i = 0$ where $\mathbf{1}_N \in \mathbb{R}^N$ is a column vector with all elements being one. Pick a matrix $U_1 \in \mathbb{R}^{N\times(N-1)}$ such that $U = \begin{bmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N & U_1 \end{bmatrix}$ is an orthogonal matrix. As a result, one has

$$U^{-1} L_i U = \begin{bmatrix} 0 & \mathbf{0}_{1\times(N-1)} \\ \mathbf{0}_{(N-1)\times 1} & H_i \end{bmatrix}$$

for some positive semidefinite matrix $H_i \in \mathbb{R}^{(N-1)\times(N-1)}$, called the *H-matrix* of the network.

Some terms about the network switching signal $\sigma(t)$ are given below.

Definition 2.2: For a piecewise constant function $\sigma(t) : [t_0, \infty) \mapsto \mathbb{P}$, the strictly increasing time sequence $\mathcal{T}_i,\ i = 0, 1, \cdots$ with $\mathcal{T}_0 = t_0$ and $\mathcal{T}_\infty = \infty$ is called a *switching sequence* of $\sigma(t)$ if, for $i = 0, 1, \cdots$,

(i) $\sigma(t)$ is constant for $t \in [\mathcal{T}_i, \mathcal{T}_{i+1})$; and
(ii) $\sigma(\mathcal{T}_i) \neq \sigma(\mathcal{T}_{i+1})$.

Each $\mathcal{T}_i$ is called a *switching instant*.

The main focus of this paper is on designing a class of decentralized controllers to achieve consensus with the switching signal and the time-varying graphs satisfying the following jointly connected assumptions.

Assumption 2.1: The switching signal $\sigma(t)$ has the switching sequence

$$t_0^0, t_0^1, \cdots, t_0^{m_0}, t_1^1, \cdots, t_1^{m_1}, t_2^1, \cdots, t_2^{m_2}, \cdots$$

for some integers $m_0, m_1, m_2, \cdots > 1$. Also, denote $t_k = t_k^0 = t_{k-1}^{m_{k-1}}$, $k = 1, 2, \cdots$. There exist constants T and τ such that

$$\begin{cases} t_{k+1} - t_k \leq T \\ t_k^{j+1} - t_k^j \geq \tau,\ j = 0, \cdots, m_k - 1 \end{cases},\ k = 0, 1, \cdots.$$

Assumption 2.2: For every time interval $[t_k, t_{k+1})$, $k = 0, 1, \cdots$, given in Assumption 2.1, the union of the graphs

$$\bar{\mathcal{G}}_k = \bigcup_{j=0}^{m_k-1} \mathcal{G}_{\sigma(t_k^j)}$$

is connected.

III. Preliminary Analysis

Each agent in (1) can be rewritten in a compact form

$$\dot{x}_i = Sx_i + Ef_i(v_i, w_i) + Eu_i$$

where

$$x_i = \begin{bmatrix} p_i \\ v_i \end{bmatrix}, \ S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ E = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The decentralized controller for each agent with the network edge weight $a_{ij}^{\sigma(t)}$ is described as follows

$$\begin{aligned} \dot{y}_i =& (S + EK)y_i + \kappa \sum_{j=1}^{N} a_{ij}^{\sigma}((y_j - y_i) + (x_i - x_j)) \\ \dot{z}_i =& \mathcal{K}_i'(v_i)Ky_i \\ u_i =& Ky_i + z_i - \mathcal{K}_i(v_i), \ i = 1, 2, \cdots, N \end{aligned} \tag{3}$$

where $y_i \in \mathbb{R}^2$, $z_i \in \mathbb{R}$, $\kappa > 0$, the function $\mathcal{K}_i(v_i)$ is to be designed later to account for the nonlinear term, $\mathcal{K}_i'(v_i) = d\mathcal{K}_i(v_i)/dv_i$, and $K \in \mathbb{R}^{1\times 2}$ is selected such that $S + EK$ is Hurwitz. In (3), y_i and z_i are the states of the dynamic controller for agent i. In particular, y_i, together with x_i, is transmitted in the network with graph $\mathcal{G}_{\sigma(t)}$ and z_i is used locally by agent i only.

With the controller (3), we have the closed-loop system of each agent as follows

$$\begin{aligned} \dot{x}_i =& Sx_i + Ef_i(v_i, w_i) + E(Ky_i + z_i - \mathcal{K}_i(v_i)) \\ \dot{y}_i =& (S + EK)y_i + \kappa \sum_{j=1}^{N} a_{ij}^{\sigma}((y_j - y_i) + (x_i - x_j)) \\ \dot{z}_i =& \mathcal{K}_i'(v_i)Ky_i, \ i = 1, 2, \cdots, N. \end{aligned} \tag{4}$$

Furthermore, (4) can be put in the compact form

$$\begin{aligned} \dot{x} =& (I_N \otimes S)x + (I_N \otimes (EK))y \\ &+ (I_N \otimes E)(f(v, w) + z - \mathcal{K}(v)) \\ \dot{y} =& (I_N \otimes (S + EK))y - \kappa(L_\sigma \otimes I_2)y + \kappa(L_\sigma \otimes I_2)x \\ \dot{z} =& \mathcal{K}'(v)(I_N \otimes K)y \end{aligned} \tag{5}$$

where $x := \mathrm{col}\{x_1, x_2, \cdots, x_N\}$, y, z, v, w in the same manner, and

$$\begin{aligned} f(v, w) :=& \mathrm{col}\{f_1(v_1, w_1), f_2(v_2, w_2), \cdots, f_N(v_N, w_N)\} \\ \mathcal{K}(v) :=& \mathrm{col}\{\mathcal{K}_1(v_1), \mathcal{K}_2(v_2), \cdots, \mathcal{K}_N(v_N)\} \\ \mathcal{K}'(v) :=& \mathrm{diag}\{\mathcal{K}_1'(v_1), \mathcal{K}_2'(v_2), \cdots, \mathcal{K}_N'(v_N)\}. \end{aligned}$$

For the convenience of further analysis, we introduce a new state variable

$$\delta = f(v, w) + z - \mathcal{K}(v).$$

Since $\dot{v} = (I_N \otimes K)y + \delta$, the dynamics of δ is calculated as follows

$$\begin{aligned} \dot{\delta} =& f'(v, w)\dot{v} + \dot{z} - \mathcal{K}'(v)\dot{v} \\ =& (f'(v, w) - \mathcal{K}'(v))\delta + f'(v, w)(I_N \otimes K)y \end{aligned}$$

where

$$\begin{aligned} f'(v, w) &= \partial f(v, w)/\partial v \\ &= \mathrm{diag}\{f_1'(v_1, w_1), \cdots, f_N'(v_N, w_N)\} \\ f_i'(v_i, w_i) &= \partial f_i(v_i, w_i)/\partial v_i. \end{aligned}$$

From the above manipulation, the closed-loop system (5) becomes

$$\begin{aligned} \dot{x} =& (I_N \otimes S)x + (I_N \otimes (EK))y + (I_N \otimes E)\delta \\ \dot{y} =& (I_N \otimes (S + EK))y - \kappa(L_\sigma \otimes I_2)y + \kappa(L_\sigma \otimes I_2)x \\ \dot{\delta} =& (f'(v, w) - \mathcal{K}'(v))\delta + f'(v, w)(I_N \otimes K)y. \end{aligned} \tag{6}$$

Remark 3.1: The system (1) includes a double-integrator model

$$\begin{aligned} \dot{p}_i &= v_i \\ \dot{v}_i &= u_i, \ i = 1, \cdots, N \end{aligned} \tag{7}$$

as a special case when the heterogeneous nonlinear function $f_i(v_i, w_i)$ vanishes. As the system matrix S for the double-integrator model is polynomially unstable, the approaches based on switched marginally stable systems in, e.g., [12], [29], are not applicable any longer. This is the essential reason that the research on linear double-integrator MASs with time-varying networks is still rare, mainly in some special situations, e.g., state-dependent switching strategy [18], sufficiently fast switching [16].

Remark 3.2: In [21], the authors proposed a dynamic controller to solve the consensus problem for a class of linear MASs with all the eigenvalues of the system matrix belonging to the closed left-half complex plane that accommodates the double-integrator model (7). In particular, the work in [21] leads to a linear closed-loop system of the form (6), but with $\delta = 0, \kappa = 1$, that is,

$$\begin{aligned} \dot{x} =& (I_N \otimes S)x + (I_N \otimes (EK))y \\ \dot{y} =& (I_N \otimes (S + EK))y - (L_\sigma \otimes I_2)y + (L_\sigma \otimes I_2)x. \end{aligned} \tag{8}$$

The problem studied in this paper is fundamentally more challenging in two aspects.

(i) The dynamic compensation for the heterogeneous function on each agent dynamics results in the additional δ-subsystem in the closed-loop system (6). Nonlinear robust design becomes critical and challenging to deal with the uncertain nonlinearities in this dynamic setup, which will be elaborated in the subsequent section.

(ii) With $\delta = 0, \kappa = 1$, the closed-loop system (8) contains an autonomous switched system,

$$\dot{\vartheta} = ((I_N \otimes S) - (L_\sigma \otimes I_2))\vartheta$$

for $\vartheta = x - y$. The approach used in [21] critically relies on convergence analysis of this switched system. While in this paper because of the nonlinearity and uncertainty in agent dynamics, the switched system is subject to external coupling from δ that does not appear in the linear homogeneous case. More specifically, the ϑ-dynamics becomes

$$\dot{\vartheta} = ((I_N \otimes S) - (L_\sigma \otimes I_2))\vartheta + (I_N \otimes E)\delta,$$

that is, there is an input δ in the dynamics of state ϑ . The analysis now relies on the novel ISS technique on switched systems to be developed in the following sections.

More analysis on the closed-loop system (6) is conducted as follows. We consider the orthogonal matrix U in Remark 2.1 and define the new states

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = (U \otimes I_2)^{-1}x, \ \bar{y} = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix} = (U \otimes I_2)^{-1}y$$

with $\bar{x}_1, \bar{y}_1 \in \mathbb{R}^2$, $\bar{x}_2, \bar{y}_2 \in \mathbb{R}^{2(N-1)}$. Then the closed-loop system (6) can be transformed into the system composed of

$$\dot{\bar{x}}_1 = S\bar{x}_1 + (EK)\bar{y}_1 + (\frac{1}{\sqrt{N}}\mathbf{1}_N^{\mathrm{T}} \otimes E)\delta \tag{9}$$

and

$$\begin{aligned}
\dot{\bar{x}}_2 =&(I_{N-1} \otimes S)\bar{x}_2 + (I_{N-1} \otimes (EK))\bar{y}_2 + (U_1^{\mathrm{T}} \otimes E)\delta \\
\dot{\bar{y}}_2 =&(I_{N-1} \otimes (S + EK))\bar{y}_2 - \kappa(H_\sigma \otimes I_2)\bar{y}_2 + \kappa(H_\sigma \otimes I_2)\bar{x}_2 \\
\dot{\delta} =&(f'(v, w) - \mathcal{K}'(v))\delta + f'(v, w)(U \otimes K)\bar{y} \\
\dot{\bar{y}}_1 =&(S + EK)\bar{y}_1.
\end{aligned} \tag{10}$$

The following proposition shows that the MAS (1) with the controller (3) achieves consensus if the equilibrium point of the system (10) is exponentially stable.

Proposition 3.1: If the equilibrium point

$$\mathrm{col}(\bar{x}_2, \bar{y}_2, \delta, \bar{y}_1) = 0$$

of the system (10) is exponentially stable, then the closed-loop MAS (6) achieves consensus as described in Definition 2.1.

Proof. Changing the variable of (9) as $\hat{x}_1 = e^{-S(t-t_0)}\bar{x}_1$ leads to

$$\dot{\hat{x}}_1 = e^{-S(t-t_0)}((EK)\bar{y}_1 + (\frac{1}{\sqrt{N}}\mathbf{1}_N^{\mathrm{T}} \otimes E)\delta)$$

and hence

$$\begin{aligned}
\hat{x}_1(t) =& \int_{t_0}^{t} e^{-S(s-t_0)}((EK)\bar{y}_1(s) \\
&+ (\frac{1}{\sqrt{N}}\mathbf{1}_N^{\mathrm{T}} \otimes E)\delta(s))ds + \hat{x}_1(t_0) \\
\triangleq& \varrho(t) + \hat{x}_1(t_0).
\end{aligned}$$

Since the equilibrium point $\mathrm{col}(\bar{x}_2, \bar{y}_2, \delta, \bar{y}_1) = 0$ of the subsystem (10) is exponentially stable and all the eigenvalues of the matrix $-S$ lie on the imaginary axis, then $\varrho(t)$ exponentially converges to a value denoted as ϱ_∞, and $\hat{x}_1(t)$ exponentially converge to a value $\hat{x}_{1\infty} = \varrho_\infty + \hat{x}_1(t_0)$. So, there exist some $\alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$\|\hat{x}_1(t) - \hat{x}_{1\infty}\| \leq \alpha_1 e^{-\alpha_2(t-t_0)}\|\hat{x}_1(t_0) - \hat{x}_{1\infty}\|.$$

As a result, we have the following inequalities

$$\begin{aligned}
&\|\bar{x}_1(t) - e^{S(t-t_0)}\hat{x}_{1\infty}\| \\
\leq&\|e^{S(t-t_0)}\|\|e^{-S(t-t_0)}(\bar{x}_1(t) - e^{S(t-t_0)}\hat{x}_{1\infty})\| \\
\leq&\|e^{S(t-t_0)}\|\|\hat{x}_1(t) - \hat{x}_{1\infty}\| \\
\leq&\alpha_1 e^{-\alpha_2(t-t_0)}\|e^{S(t-t_0)}\|\|\hat{x}_1(t_0) - \hat{x}_{1\infty}\|.
\end{aligned}$$

Since $e^{S(t-t_0)} = \begin{bmatrix} 1 & t-t_0 \\ 0 & 1 \end{bmatrix}$, then $\|e^{S(t-t_0)}\|$ increases polynomially with respect to t. There exist some $\bar{\alpha}_1 > 0$ and $\bar{\alpha}_2 > 0$ such that

$$\|e^{S(t-t_0)}\| \leq \frac{\bar{\alpha}_1}{\alpha_1} e^{(\alpha_2 - \bar{\alpha}_2)(t-t_0)}.$$

Therefore, we have

$$\|\bar{x}_1(t) - e^{S(t-t_0)}\hat{x}_{1\infty}\| \leq \bar{\alpha}_1 e^{-\bar{\alpha}_2(t-t_0)}\|\bar{x}_1(t_0) - \hat{x}_{1\infty}\|.$$

It is ready to see that the solution of (9) exponentially converges to a system with solution $x_0(t) = e^{S(t-t_0)}\hat{x}_{1\infty}$, that is, $\lim_{t\to\infty}[\bar{x}_1(t) - x_0(t)] = 0$. The fact

$$x(t) = (U \otimes I_2)\bar{x} = \frac{1}{\sqrt{N}}(\mathbf{1}_N \otimes I_2)\bar{x}_1(t) + (U_1 \otimes I_2)\bar{x}_2(t)$$

yields

$$\lim_{t\to\infty}\left[x(t) - (\mathbf{1}_N \otimes I_2)\frac{1}{\sqrt{N}}x_0(t)\right] = 0.$$

Let $x_0 = \mathrm{col}\{p_0, v_0\}$. Then consensus of the MAS (1) with the virtual reference (2) is obtained. ■

What is left is to study the exponential stability of the system (10). It is noted that the system (10) can be regarded as a switched system of which no subsystem is assumed to be stable. Then how to select a proper $\mathcal{K}_i(v_i)$ such that the system (10) is exponentially stable under the switching signal $\sigma(t)$ will be elaborated in next sections using the approach in terms of ISS of switched systems.

IV. Input-to-State Stability of Switched Systems

We first consider the ISS property of a class of linear switched control systems:

$$\dot{\xi}(t) = A_{\sigma(t)}\xi(t) + B_{\sigma(t)}u(t) \tag{11}$$

where $\xi(t) : [t_0, \infty) \mapsto \mathbb{R}^n$ is the state and $u(t) : [t_0, \infty) \mapsto \mathbb{R}^m$ the input. Here t_0 represents the initial time. The matrices $A_{\sigma(t)} \in \mathbb{R}^{n\times n}$ and $B_{\sigma(t)} \in \mathbb{R}^{n\times m}$ depend on the switching signal $\sigma(t) : [t_0, \infty) \mapsto \mathbb{P}$ for $\mathbb{P} := \{1, 2, \cdots, M\}$ that is assumed to be a continuous-time piecewise constant function. The equation (11) with a fixed $\sigma(t) = i$ is called a subsystem, more specifically, the i-th subsystem, for every $i \in \mathbb{P}$. In other words, the switched system (11) consists of $M > 1$ subsystems obeying the switching signal $\sigma(t)$.

Suppose the switching signal $\sigma(t)$ satisfies Assumption 2.1 and the switched system (11) satisfies the following assumption.

Assumption 4.1: Every subsystem matrix A_i, $i \in \mathbb{P}$, is negative semidefinite. Furthermore, for every time interval $[t_k, t_{k+1})$, $k = 0, 1, \cdots$, given in Assumption 2.1, the switched subsystem matrices satisfy

$$\bigcap_{j=0}^{m_k-1} \ker\left(A_{\sigma(t_k^j)}\right) = \emptyset \tag{12}$$

where $\ker(\cdot)$ denotes the kernel of a matrix.

The main objective of this section is to study the ISS property of the system (11) for an arbitrary switching signal $\sigma(t)$ satisfying Assumptions 2.1 and 4.1. For this purpose, the concept of ISS is rigorously given below,

Definition 4.1: The switched system (11) is said to be input-to-state stable (ISS) with the state ξ and the input u, if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ, independent of t_0, such that for any initial state ξ_0 and any input function $u \in L_\infty^m$, the solution $\xi(t)$ satisfies[1]

$$\|\xi(t)\| \leq \beta(\|\xi(t_0)\|, t - t_0) + \gamma(\|u_{[t_0,t]}\|),\ t \geq t_0.$$

Moreover, it is said to be exponentially ISS if the class $\mathcal{KL}$ function takes an exponential form $\beta(r, s) = \varkappa r e^{-\alpha s}$ for $\varkappa, \alpha > 0$.

Before we investigate the ISS property of the system (11), we first study the simpler scenario without considering the input u. In particular, the switched system (11) without the input u becomes

$$\dot{\xi}(t) = A_{\sigma(t)}\xi(t). \tag{13}$$

For convergence analysis of the switched system (13), we construct a Lyapunov-like function as follows,

$$V(t) = \xi^{\mathrm{T}}(t)\xi(t). \tag{14}$$

[1] We use the notation L_∞^m to denote the set of all piecewise continuous bounded functions $u : [t_0, \infty) \mapsto \mathbb{R}^m$. The supremum norm of the truncation of $u(t)$ in $[t_0, t]$ is denoted as $\|u_{[t_0,t]}\| = \sup_{t_0\leq\tau\leq t}\|u(\tau)\|$.

Consider every subsystem matrix $A_i,\ i \in \mathbb{P}$. It is easy to see that, along the trajectory of each subsystem of (13),

$$\dot{V}(t) = \xi^{\mathrm{T}}(t)(A_i + A_i^{\mathrm{T}})\xi(t) \leq 0. \tag{15}$$

The property of the function $V(t)$ along the switched system (13) is given in the following lemma that plays an important role in the subsequent analysis. It is noted that a relevant property has been studied in [12] for a special class of systems when it was hidden in the consensus analysis of MASs.

Lemma 4.1: Consider the switched system (13) with the switching signal $\sigma(t)$ satisfying Assumptions 2.1 and 4.1. There exists $0 < \rho < 1$ such that, for every time interval $[t_k, t_{k+1}), k = 0, 1, \cdots$, the function $V(t)$ defined in (14) satisfies

$$V(t_{k+1}) \leq \rho V(t_k). \tag{16}$$

Now it is ready to have the main statement on the ISS property of the switched system (11) as follows. The proof is ignored due to the space limit.

Theorem 4.1: The switched system (11) with the switching signal $\sigma(t)$ satisfying Assumptions 2.1 and 4.1 is exponentially ISS, in particular,

$$\|\xi(t)\| \leq \beta(\|\xi(t_0)\|, t - t_0) + \gamma(\|u_{[t_0,t]}\|),\ \forall t \geq t_0 \tag{17}$$

for a $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ. Moreover, the gain function γ can be explicitly given as

$$\gamma(s) = B\frac{2-\sqrt{\rho}}{1-\sqrt{\rho}}Ts$$

for $B = \max_{i\in\mathbb{P}}\{\|B_i\|\}$ and ρ given in Lemma 4.1.

Next, we can extend the ISS property to a class of switched systems of more complicated interconnected structure, i.e.,

$$\dot{\xi} = \begin{bmatrix} A_\sigma & I_n \\ 0 & A_\sigma \end{bmatrix}\xi + \begin{bmatrix} 0 \\ B_\sigma \end{bmatrix}u \tag{18}$$

where $\xi(t) : [t_0, \infty) \mapsto \mathbb{R}^{2n}$ is the state, $u(t) : [t_0, \infty) \mapsto \mathbb{R}^m$ the input, and $I_n \in \mathbb{R}^{n\times n}$ an identity matrix. The other notation is defined as in (11). We consider the system with the switching signal $\sigma(t)$ satisfying Assumptions 2.1 and 4.1. It is worth mentioning that, we assume that every matrix $A_i,\ i \in \mathbb{P}$, is negative semidefinite (and hence marginally stable) in Assumption 4.1, however, the actual system matrix of (18) now becomes

$$\begin{bmatrix} A_i & I_n \\ 0 & A_i \end{bmatrix}$$

that is asymmetric and polynomially unstable. The following corollary shows that the switched system is still ISS with the proper switching signal even if all of its subsystems are unstable.

Corollary 4.1: The switched system (18) with the switching signal $\sigma(t)$ satisfying Assumptions 2.1 and 4.1 is ISS.

V. Solvability of Consensus Problem

In this section, we will study the exponential stability of the switched system (10) under the switched signal $\sigma(t)$ using the approach proposed in Section IV and the small gain theorem. Firstly, the connection between switching networks in the multi-agent setup in Section II and the results on switched systems in Section IV is stated in the following lemma.

Lemma 5.1: Consider the switching network with graph $\mathcal{G}_{\sigma(t)}$ and the switching signal $\sigma(t)$ satisfying Assumption 2.1. Let $H_{\sigma(t)}$ be the H-matrix of the network and $A_{\sigma(t)} = -H_{\sigma(t)}$. The graphs satisfy Assumption 2.2 if and only if the switched system $\dot{\xi} = A_{\sigma(t)}\xi$ satisfies Assumption 4.1.

Proof. It is noted that $A_\sigma = -H_\sigma$ is symmetric and negative semidefinite. The graphs satisfy Assumption 2.2 if and only if the H-matrices has the corresponding property $-\sum_{j=0}^{m_k-1} H_{\sigma(t_k^j)} < 0$, i.e., $\sum_{j=0}^{m_k-1} A_{\sigma(t_k^j)} < 0$ is negative definite. The negative definite property of $\sum_{j=0}^{m_k-1} A_{\sigma(t_k^j)} < 0$ is equivalent to that $\xi^{\mathrm{T}}(\sum_{j=0}^{m_k-1} A_{\sigma(t_k^j)})\xi < 0$ for all $\xi \neq \mathbf{0} \in \mathbb{R}^{N-1}$. Finally, $\xi^{\mathrm{T}}(\sum_{j=0}^{m_k-1} A_{\sigma(t_k^j)})\xi < 0$ for all $\xi \neq \mathbf{0} \in \mathbb{R}^{N-1}$ if and only if $\bigcap_{j=0}^{m_k-1} \ker\left(A_{\sigma(t_k^j)}\right) = \emptyset$, that is, $\dot{\xi} = A_{\sigma(t)}\xi$ satisfies Assumption 4.1. The proof is completed.

The final argument is further explained as follows. (i) If there is $\xi \neq \mathbf{0}$ such that $\xi^{\mathrm{T}}(\sum_{j=0}^{m_k-1} A_{\sigma(t_k^j)})\xi = 0$, then $\xi^{\mathrm{T}}A_{\sigma(t_k^j)}\xi = 0$ for all j, and hence $\xi \in \bigcap_{j=0}^{m_k-1} \ker\left(A_{\sigma(t_k^j)}\right) \neq \emptyset$. (ii) If $\xi \neq \mathbf{0} \in \bigcap_{j=0}^{m_k-1} \ker\left(A_{\sigma(t_k^j)}\right) \neq \emptyset$, then $\xi^{\mathrm{T}}A_{\sigma(t_k^j)}\xi = 0$ for all j, and hence $\xi^{\mathrm{T}}(\sum_{j=0}^{m_k-1} A_{\sigma(t_k^j)})\xi = 0$. ■

In terms of the system (10), let $\chi = \bar{x}_2 - \bar{y}_2$, then the system (10) can be rewritten into

$$\begin{aligned}
\dot{\chi} =& ((I_{N-1}\otimes S) - \kappa(H_\sigma \otimes I_2))\chi + (U_1^{\mathrm{T}} \otimes E)\delta \\
\dot{\bar{x}}_2 =& (I_{N-1}\otimes (S+EK))\bar{x}_2 - (I_{N-1}\otimes (EK))\chi + (U_1^{\mathrm{T}}\otimes E)\delta \\
\dot{\delta} =& (f'(v,w) - \mathcal{K}'(v))\delta + f'(v,w)(U\otimes K)\Theta_1\bar{y}_1 \\
&+ f'(v,w)(U\otimes K)\Theta_2\begin{bmatrix} \bar{x}_2 \\ \chi \end{bmatrix} \\
\dot{\bar{y}}_1 =& (S+EK)\bar{y}_1
\end{aligned} \tag{19}$$

where

$$\Theta_1 = \begin{bmatrix} I_2 \\ \mathbf{0}_{2(N-1)\times 2} \end{bmatrix},\ \Theta_2 = \begin{bmatrix} \mathbf{0}_{2\times 2(N-1)} & \mathbf{0}_{2\times 2(N-1)} \\ I_{2(N-1)} & -I_{2(N-1)} \end{bmatrix}.$$

The system (19) can be regarded as interconnection of four subsystems denoted as $\Sigma_1(\chi)$, $\Sigma_2(\bar{x}_2)$, $\Sigma_3(\delta)$, and $\Sigma_4(\bar{y}_1)$ according to their states. To verify the exponential stability of the system (19), the first task is to show that each subsystem in (19) is (exponentially) ISS. Then by the small gain theorem, the overall system (19) is exponentially stable if the interconnected gain functions satisfy a certain small gain condition.

The subsystems $\Sigma_2(\bar{x}_2)$, $\Sigma_3(\delta)$, $\Sigma_4(\bar{y}_1)$ are purely continuous-time systems that are independent of the switching signals $\sigma(t)$. Their ISS properties can achieved using nonlinear robust design. However, it is noted that the subsystem $\Sigma_1(\chi)$ is a switched system subject to the external input δ.

In this paper, it is observed that $((I_{N-1}\otimes S) - \kappa(H_\sigma \otimes I_2))$ is unstable and none of the switched subsystems of $\Sigma_1(\chi)$ is ISS with respect to δ. Therefore the existing results [23], [30], [31], [32] cannot apply to the present situation. The design now relies on the new approach developed in the previous section.

Based on the above analysis, the result on exponential stability of the switched system (19) under the switching signal $\sigma(t)$ and the consensus conditions for the MAS (1) with the controller (3) under the jointly connected topology are stated in the following theorem.

Theorem 5.1: Consider the MAS (1) with the controller (3) in the switching network with graphs $\mathcal{G}_{\sigma(t)}$. Suppose the switching signal $\sigma(t)$ satisfies Assumptions 2.1 and 2.2. In (3), let K be selected such that $S + EK$ is Hurwitz and $P > 0$ such that

$$-Q = P(S+EK) + (S+EK)^{\mathrm{T}}P + 2\epsilon PEE^{\mathrm{T}}P < 0$$

for some $\epsilon > 0$. Also, let

$$\mathcal{K}'(v) > \mathcal{F}(v) + \varepsilon\mathcal{F}(v)\mathcal{F}(v) + \Lambda$$

for some $\varepsilon > 0$ and diagonal matrix $\Lambda > 0$. Then, the system (19) satisfies the following four properties.

(1) The system $\Sigma_1(\chi)$ is ISS with respect to input δ, in particular,

$$\|\chi(t)\| \leq \beta_1(\|\chi(t_0)\|, t-t_0) + \gamma_1\|\delta_{[t_0,t]}\|, \ \forall t \geq t_0,$$

for some class $\mathcal{KL}$ function β_1 and constant $\gamma_1 > 0$.

(2) The system $\Sigma_2(\bar{x}_2)$ is ISS with respect to input $\mathrm{col}\{\chi, \delta\}$, in particular,

$$\|\bar{x}_2(t)\| \leq \beta_2(\|\bar{x}_2(t_0)\|, t-t_0) + \gamma_2^{\chi}\|\chi_{[t_0,t]}\| + \gamma_2^{\delta}\|\delta_{[t_0,t]}\|, \ \forall t \geq t_0,$$

for some class $\mathcal{KL}$ function β_2 and two constants[2]

$$\gamma_2^{\chi} > \sqrt{\frac{\lambda_{\max}(P)\lambda_{\max}(K^{\mathrm{T}}K)}{\epsilon\lambda_{\min}(P)\lambda_{\min}(Q)}},$$
$$\gamma_2^{\delta} > \sqrt{\frac{\lambda_{\max}(P)}{\epsilon\lambda_{\min}(P)\lambda_{\min}(Q)}}.$$

(3) The system $\Sigma_3(\delta)$ is ISS with respect to input $\mathrm{col}\{\bar{y}_1, \bar{x}_2, \chi\}$, in particular, with $\psi = \mathrm{col}\{\bar{x}_2, \chi\}$,

$$\|\delta(t)\| \leq \beta_3(\|\delta(t_0)\|, t-t_0) + \gamma_3^{\bar{y}_1}\|\bar{y}_{1[t_0,t]}\| + (\gamma_3^{\psi}/\sqrt{\varepsilon})\|\psi_{[t_0,t]}\|, \ \forall t \geq t_0,$$

for some class $\mathcal{KL}$ function β_3, some constant $\gamma_3^{\bar{y}_1} > 0$, and a constant

$$\gamma_3^{\psi} > \sqrt{\frac{\lambda_{\max}(K^{\mathrm{T}}K)}{2\lambda_{\min}(\Lambda)}}.$$

(4) The system $\Sigma_4(\bar{y}_1)$ is exponentially stable.

Moreover, if the control parameter ε is selected such that

$$\sqrt{\varepsilon} > 2\gamma_3^{\psi}\max\{2\gamma_2^{\chi}\gamma_1, 2\gamma_2^{\delta}, 2\gamma_1\}, \tag{20}$$

then the closed-loop MAS achieves consensus as described in Definition 2.1.

Proof. The proofs of the four properties and the moreover part are given below in order.

Property (1):

First of all, the system $\Sigma_1(\chi)$ can be rewritten as

$$\dot{\bar{\chi}} = \begin{bmatrix} -\kappa H_\sigma & I_{N-1} \\ 0 & -\kappa H_\sigma \end{bmatrix}\bar{\chi} + \begin{bmatrix} 0 \\ U_1^{\mathrm{T}} \end{bmatrix}\delta \tag{21}$$

where $\bar{\chi} = \bar{I}\chi = \mathrm{col}\{\bar{\chi}_1, \bar{\chi}_2\}$, $\bar{\chi}_1, \bar{\chi}_2 \in \mathbb{R}^{N-1}$, $\bar{I} = \mathrm{col}\{\Xi_1, \Xi_N, \Xi_2, \Xi_{N+1}, \cdots \Xi_{N-1}, \Xi_{2(N-1)}\}$ and $\Xi_i \in \mathbb{R}^{1\times 2(N-1)}, i = 1, \cdots, 2(N-1)$ with all elements being zero except the $i-$th element being one. It is easy to see that $\bar{I}$ is an elementary matrix satisfying $\bar{I}^{-1} = \bar{I}$.

Obviously, the system $\Sigma_1(\chi)$ is equivalent to the system (21) that is of the form (18). It is ready to verify ISS of the linear switched system (21) using Corollary 4.1 and Lemma 5.1. More specifically, there exist some $\mathcal{KL}$ function β_1 and constant γ_1 such that

$$\|\bar{\chi}(t)\| \leq \beta_1(\|\bar{\chi}(t_0)\|, t-t_0) + \gamma_1\|\delta_{[t_0,t]}\|, \ \forall t \geq t_0. \tag{22}$$

In particular, the constant gain function γ_1 can be explicitly calculated using Corollary 4.1 and Theorem 4.1. Property (1) is thus proved by noting $\|\chi(t)\| = \|\bar{\chi}(t)\|$.

Property (2):

Construct a Lyapunov function for the system $\Sigma_2(\bar{x}_2)$ as follows

$$V_2(\bar{x}_2) = \bar{x}_2^{\mathrm{T}}(I_{N-1}\otimes P)\bar{x}_2$$

that satisfies

$$\lambda_{\min}(P)\|\bar{x}_2\|^2 \leq V_2(\bar{x}_2) \leq \lambda_{\max}(P)\|\bar{x}_2\|^2.$$

The derivative of $V_2(\bar{x}_2)$ along the trajectory of $\Sigma_2(\bar{x}_2)$ is

$$\begin{aligned}\dot{V}_2(\bar{x}_2) =& 2\bar{x}_2^{\mathrm{T}}(I_{N-1}\otimes P)\dot{\bar{x}}_2 \\ =& -\bar{x}_2^{\mathrm{T}}(I_{N-1}\otimes \bar{Q})\bar{x}_2 - 2\bar{x}_2^{\mathrm{T}}(I_{N-1}\otimes (PEK))\chi \\ &+ 2\bar{x}_2^{\mathrm{T}}(U_1^{\mathrm{T}}\otimes (PE))\delta\end{aligned}$$

for $-\bar{Q} = P(S+EK) + (S+EK)^{\mathrm{T}}P$.

It is noted that

$$\begin{aligned}&-2\bar{x}_2^{\mathrm{T}}(I_{N-1}\otimes (PEK))\chi \\ =& -2\bar{x}_2^{\mathrm{T}}(I_{N-1}\otimes (PE))(I_{N-1}\otimes K)\chi \\ \leq& \epsilon\bar{x}_2^{\mathrm{T}}(I_{N-1}\otimes (PEE^{\mathrm{T}}P))\bar{x}_2 + \frac{1}{\epsilon}\chi^{\mathrm{T}}(I_{N-1}\otimes (K^{\mathrm{T}}K))\chi\end{aligned}$$

and

$$2\bar{x}_2^{\mathrm{T}}(U_1^{\mathrm{T}}\otimes (PE))\delta \leq \epsilon\bar{x}_2^{\mathrm{T}}(I_{N-1}\otimes (PEE^{\mathrm{T}}P))\bar{x}_2 + \frac{1}{\epsilon}\delta^{\mathrm{T}}\delta$$

for $\epsilon > 0$. For $-Q = -\bar{Q} + 2\epsilon PEE^{\mathrm{T}}P < 0$, one has

$$\dot{V}_2(\bar{x}_2) \leq -\lambda_{\min}(Q)\|\bar{x}_2\|^2 + \frac{1}{\epsilon}\lambda_{\max}(K^{\mathrm{T}}K)\|\chi\|^2 + \frac{1}{\epsilon}\|\delta\|^2.$$

Therefore, the system $\Sigma_2(\bar{x}_2)$ is ISS viewing $\bar{x}_2$ as state and $\mathrm{col}\{\chi, \delta\}$ as input. In particular, there exist some class $\mathcal{KL}$ function β_2 such that

$$\|\bar{x}_2(t)\| \leq \beta_2(\|\bar{x}_2(t_0)\|, t-t_0) + \gamma_2^{\chi}\|\chi_{[t_0,t]}\| + \gamma_2^{\delta}\|\delta_{[t_0,t]}\|, \ \forall t \geq t_0, \tag{23}$$

for the two constants $\gamma_2^{\chi}, \gamma_2^{\delta}$ given in Property (2).

Property (3):

Construct a Lyapunov function for the system $\Sigma_3(\delta)$ as follows

$$V_3(\delta) = \delta^{\mathrm{T}}\delta.$$

The derivative of $V_3(\delta)$ along the trajectory of $\Sigma_3(\delta)$ is

$$\begin{aligned}\dot{V}_3(\delta) =& 2\delta^{\mathrm{T}}\dot{\delta} \\ =& 2\delta^{\mathrm{T}}(f'(v,w) - \mathcal{K}'(v))\delta + 2\delta^{\mathrm{T}}f'(v,w)(U\otimes K)\Theta_1\bar{y}_1 \\ &+ 2\delta^{\mathrm{T}}f'(v,w)(U\otimes K)\Theta_2\psi\end{aligned}$$

where $\psi = \mathrm{col}\{\bar{x}_2, \chi\}$.

As

$$\mathcal{K}'(v) > \mathcal{F}(v) + \varepsilon\mathcal{F}(v)\mathcal{F}(v) + \Lambda,$$

there exists $\tilde{\varepsilon} > 0$ such that

$$\mathcal{K}'(v) \geq \mathcal{F}(v) + (\tilde{\varepsilon} + \varepsilon)\mathcal{F}(v)\mathcal{F}(v) + \Lambda.$$

It is noted that

$$\begin{aligned}&2\delta^{\mathrm{T}}f'(v,w)(U\otimes K)\Theta_1\bar{y}_1 \\ \leq& 2\tilde{\varepsilon}\delta^{\mathrm{T}}f'(v,w)f'(v,w)\delta + \frac{1}{2\tilde{\varepsilon}}\bar{y}_1^{\mathrm{T}}\Theta_1^{\mathrm{T}}(U\otimes K)^{\mathrm{T}}(U\otimes K)\Theta_1\bar{y}_1 \\ \leq& 2\tilde{\varepsilon}\delta^{\mathrm{T}}f'(v,w)f'(v,w)\delta + \frac{1}{2\tilde{\varepsilon}}\bar{y}_1^{\mathrm{T}}(K^{\mathrm{T}}K)\bar{y}_1\end{aligned}$$

and

$$\begin{aligned}&2\delta^{\mathrm{T}}f'(v,w)(U\otimes K)\Theta_2\psi \\ \leq& 2\varepsilon\delta^{\mathrm{T}}f'(v,w)f'(v,w)\delta + \frac{1}{2\varepsilon}\psi^{\mathrm{T}}\Theta_2^{\mathrm{T}}(U\otimes K)^{\mathrm{T}}(U\otimes K)\Theta_2\psi \\ \leq& 2\varepsilon\delta^{\mathrm{T}}f'(v,w)f'(v,w)\delta + \frac{1}{2\varepsilon}\psi^{\mathrm{T}}((\Upsilon\otimes I_{N-1})\otimes (K^{\mathrm{T}}K))\psi\end{aligned}$$

[2]For a real symmetric matrix A, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote its maximum and minimum eigenvalues, respectively.

with $\Upsilon = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ for $\tilde{\varepsilon}, \varepsilon > 0$. Furthermore, recall the condition $|\partial f_i(v_i, w_i)/\partial v_i| \leq \mathcal{F}_i(v_i)$. Then using $f'(v,w) \leq \mathcal{F}(v)$ and $f'(v,w)f'(v,w) \leq \mathcal{F}(v)\mathcal{F}(v)$, we have

$$\begin{aligned}\dot{V}_3(\delta) \leq & 2\delta^{\mathrm{T}}(\mathcal{F}(v) - \mathcal{K}'(v))\delta + 2(\tilde{\varepsilon} + \varepsilon)\delta^{\mathrm{T}}\mathcal{F}(v)\mathcal{F}(v)\delta \\ & + \frac{1}{2\tilde{\varepsilon}}\bar{y}_1^{\mathrm{T}}(K^{\mathrm{T}}K)\bar{y}_1 + \frac{1}{2\varepsilon}\psi^{\mathrm{T}}((\Upsilon \otimes I_{N-1}) \otimes (K^{\mathrm{T}}K))\psi\end{aligned}$$

and

$$\dot{V}_3(\delta) \leq -2\lambda_{\min}(\Lambda)\|\delta\|^2 + \frac{1}{2\tilde{\varepsilon}}\lambda_{\max}(K^{\mathrm{T}}K)\|\bar{y}_1\|^2 + \frac{1}{\varepsilon}\lambda_{\max}(K^{\mathrm{T}}K)\|\psi\|^2.$$

Therefore the system $\Sigma_3(\delta)$ is ISS viewing δ as state and $\mathrm{col}\{\bar{y}_1, \psi\}$ as input. In particular, there exist some class $\mathcal{KL}$ function β_3 such that

$$\begin{aligned}\|\delta(t)\| \leq & \beta_3(\|\delta(t_0)\|, t - t_0) + \gamma_3^{\bar{y}_1}\|\bar{y}_{1[t_0,t]}\| \\ & + (\gamma_3^{\psi}/\sqrt{\varepsilon})\|\psi_{[t_0,t]}\|, \ \forall t \geq t_0,\end{aligned} \tag{24}$$

for the constant γ_3^{ψ} given in Property (3) and

$$\gamma_3^{\bar{y}_1} = \sqrt{\frac{\lambda_{\max}(K^{\mathrm{T}}K)}{4\tilde{\varepsilon}\lambda_{\min}(\Lambda)}}.$$

Property (4):

As $S + EK$ is Hurwitz, the proof is straightforward. More specifically, we construct a Lyapunov function for $\Sigma_4(\bar{y}_1)$ as follows

$$V_4(\bar{y}_1) = \bar{y}_1^{\mathrm{T}}P\bar{y}_1.$$

It is easy to see that

$$\lambda_{\min}(P)\|\bar{y}_1\|^2 \leq V_4(\bar{y}_1) \leq \lambda_{\max}(P)\|\bar{y}_1\|^2$$

and

$$\begin{aligned}\dot{V}_4(\bar{y}_1) = & 2\bar{y}_1^{\mathrm{T}}P\dot{\bar{y}}_1 = -\bar{y}_1^{\mathrm{T}}Q\bar{y}_1 \\ \leq & -\lambda_{\min}(Q)\|\bar{y}_1\|^2 \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}V_4(\bar{y}_1) \triangleq -\alpha_4 V_4(\bar{y}_1).\end{aligned}$$

Then, we have

$$V_4(\bar{y}_1(t)) \leq e^{-\alpha_4(t-t_0)}V_4(\bar{y}_1(t_0)),$$

that is,

$$\|\bar{y}_1(t)\|^2 \leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}e^{-\alpha_4(t-t_0)}\|\bar{y}_1(t_0)\|^2.$$

Therefore, exponential stability of the system $\Sigma_4(\bar{y}_1)$ is guaranteed and there exists a $\mathcal{KL}$ function β_4 such that

$$\|\bar{y}_1(t)\| \leq \beta_4(\|\bar{y}_1(t_0)\|, t - t_0), \ \forall t \geq t_0. \tag{25}$$

Moreover part:

Let $\varphi = \mathrm{col}\{\delta, \bar{y}_1\}$. Use $\|\delta_{[t_0,t]}\| \leq \|\varphi_{[t_0,t]}\|$ and recall (22) and (23). Then we have

$$\begin{aligned}\|\chi(t)\| \leq & \beta_1(\|\chi(t_0)\|, t - t_0) + \gamma_1\|\varphi_{[t_0,t]}\| \\ \|\bar{x}_2(t)\| \leq & \beta_2(\|\bar{x}_2(t_0)\|, t - t_0) + \gamma_2^{\chi}\|\chi_{[t_0,t]}\| + \gamma_2^{\delta}\|\varphi_{[t_0,t]}\|.\end{aligned}$$

By the small gain theorem, the combined system viewing $\psi = \mathrm{col}\{\bar{x}_2, \chi\}$ as state and φ as input is ISS. In particular, there exists some class $\mathcal{KL}$ function $\hat{\beta}_1$ such that

$$\|\psi(t)\| \leq \hat{\beta}_1(\|\psi(t_0)\|, t - t_0) + \gamma^{\varphi}\|\varphi_{[t_0,t]}\|, \ \forall t \geq t_0, \tag{26}$$

for a constant $\gamma^{\varphi} = \max\{2\gamma_2^{\chi}\gamma_1, 2\gamma_2^{\delta}, 2\gamma_1\}$.

Similarly, recall (24) and (25). We have the combined system viewing $\varphi = \mathrm{col}\{\delta, \bar{y}_1\}$ as state and ψ as input is ISS. In particular, there exists some class $\mathcal{KL}$ function $\hat{\beta}_2$ such that

$$\|\varphi(t)\| \leq \hat{\beta}_2(\|\varphi(t_0)\|, t - t_0) + \gamma^{\psi}(\|\psi_{[t_0,t]}\|), \ \forall t \geq t_0, \tag{27}$$

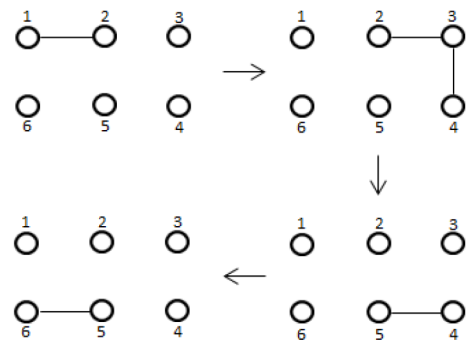

Fig. 1. Switching topologies with $N = 6$ and $M = 4$.

for a constant $\gamma^{\psi} = 2\gamma_3^{\psi}/\sqrt{\varepsilon}$.

The condition (20) is equivalent to

$$\gamma^{\psi}\gamma^{\varphi} < 1. \tag{28}$$

From (26) and (27) and the small gain condition (20), we can conclude that the equilibrium point $\mathrm{col}(\bar{x}_2, \bar{y}_2, \delta, \bar{y}_1) = 0$ of the system (10) is exponentially stable (noting all the class $\mathcal{KL}$ functions in the aforementioned four properties take exponential form). The proof is thus completed by applying Proposition 3.1. ■

VI. Numerical Simulation

In this section, an MAS consisting of six agents of dynamics (1) is simulated. Assume the nonlinearity is $f_i(v_i, w_i) = w_i v_i^3$ with w_i arbitrarily selected in the interval $[-1, 1]$. Then $\mathcal{F}_i(v_i) = 3v_i^2$ can be chosen. The time-varying topology $\mathcal{G}_{\sigma(t)}$ is assumed to be periodic and the piecewise constant switching signal $\sigma(t)$ belongs to the set $\mathbb{P} = \{1, 2, 3, 4\}$. The corresponding undirected topologies $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$, are depicted in Fig. 1 clock-wisely from the upper left. Note that none of the topologies is connected but they satisfy Assumption 2.2, that is, $\bigcup_{i=1}^4 \mathcal{G}_i$ is connected. A periodically switching sequence is considered in the simulation. The parameters and functions $K, \kappa, \mathcal{K}_i(v_i)$, $i = 1, 2, \cdots, N$ in the control protocol (3) can be explicitly designed following the approach developed in this paper.

The agent trajectories under each unconnected topology are checked with obviously no consensus achieved. When the topology is switching periodically, the velocity and position trajectories are plotted in Fig. 2 which shows that both the velocities and the positions of all the agents achieve consensus as expected.

VII. Conclusion

In this paper, we have solved the consensus problem for a class of nonlinear, heterogeneous, and uncertain MASs in switching networks. Achievement of consensus is guaranteed under the so-called jointly connected assumption, which does not require measurement of agent state or sufficiently fast switching. Success of the design relies on a novel approach for ISS analysis of switched systems. The consensus problem of an MAS is based on stability of interconnection of a switched system and a nonlinear dynamic compensator.

The framework in this paper is set up for undirected graphs. The fundamental technical challenges for directed graphs in the scenarios under investigation are explained below for future work.

(i) The convergence analysis technique used in this paper relies on the empty intersection condition for the kernels of switched system matrices (represented by $-H_i$; see Assumption 4.1). As shown in Lemma 5.1, for undirected graphs, this condition is equivalent to the

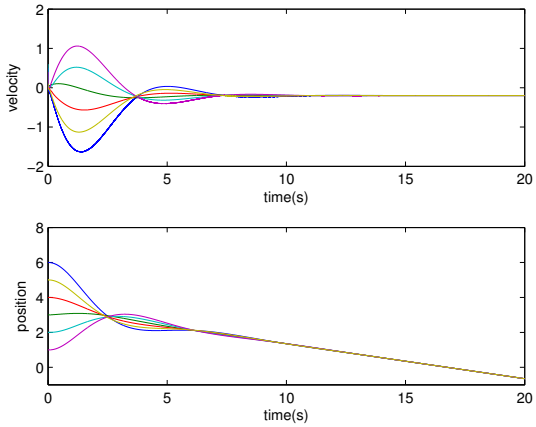

Fig. 2. Profile of state trajectories under jointly connected topology.

jointly connected condition. However, for directed graphs, a non-convergent network not satisfying the jointly connected condition may still satisfy the empty intersection condition. Therefore, the convergence analysis technique based on the empty intersection condition is unlikely to be applicable for directed graphs.

(ii) It is noted that the results in [12], [14], [15], etc. were established on undirected graphs that induce symmetric H-matrices. The resulting switched system may have an asymmetric matrix A_i that however has a special structure of, for example, $A_i = I \otimes A - H_i \otimes BK$ with symmetric H_i. Therefore, a common Lyapunov-like function can be constructed for the switched system. One contribution of this paper is to follow this research line and develop the technique to an ISS setting that is critical for convergence analysis of interconnected nonlinear dynamics. For directed graphs, the H-matrix is asymmetric and it is unlikely to find a common Lyapunov-like function for the resulting switched system. Another research line in, e.g., [10], [21], offers a set-value Lyapunov function that is applicable for this situation. So far, we have yet to see a clue for extending the mechanism to an ISS setting with an explicit gain.

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